Beating: a Neglected Pattern Formation Mechanism in Nonlinear Systems

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Abstract

In this essay, I have brought examples of the properties of beating phenomena that are not fully explored in the existing literature. Robustness of the beating under amplitude and phase noise, multidimensional pattern formation and passing beating into nonlinear systems are the phenomena I have explored here. Moreover, I have discuss how beating can lead to existence of different time scales in the brain and how it can possibly account for scaler variability in time perception.

1 Beating via examples

Since 2011, I have been working on Voltage Sensitive Dye (VSD) imaging data of the mouse brain. Alongside developing various motion tracking, compressed sensing and sparse recovery algorithms for VSD image sequences (my BSc and MSc projects), I would run various data analysis experiments on the data to understand the physiological models behind the complex cortical activity patterns.

In one of the experiments, I examined frequency contents of different areas of the neocortex during spontaneous activity in anaesthetised mouse. The observations suggested a model for ultra-slow waves. In fact, despite α-rhythm-like oscillations which are quite periodic (see Figure 1), in long term, one can see a slow wave of increase and decrease in the activity. In examining the frequency spectrum of different areas of the neocortex, I observed a slight difference in frequency response for some neighbouring areas.

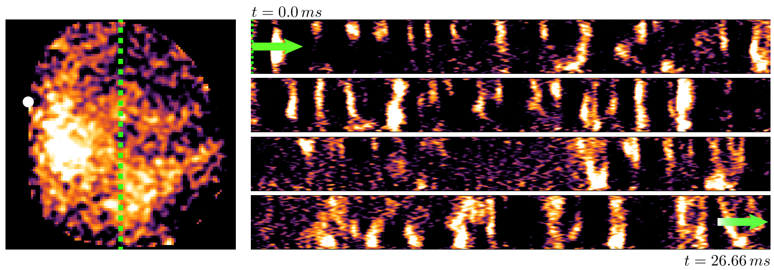

Figure 1: Cortical Spontaneous Activity space-time map captured by VSD imaging (RH1-691 dye) . Note the slow up-down or beating phenomena over time. The white dot demonstrates the bregma.

such a difference can be considered as the effect of noise or nonlinear distortions. But it can also account for a phenomenon known as beating. Here an acoustic analogy can be helpful: Consider two interfering monotone sources with frequencies ω_1 and ω_2. The resulting signal can be written as:

$$\cos(\omega_1 t) + \cos(\omega_2 t) = \underbrace{2\cos(\frac{\omega_1+\omega_2}{2}t)}_{\text{crrier (HF)}}\underbrace{\cos(\frac{\omega_1-\omega_2}{2}t)}_{\text{envelope (LF)}}. \quad (1)$$

As we see, the output signal can be thought of a low frequency signal, modulated with a high-frequency carrier. This mathematical observation accounts for a well-known physical phenomena called *Beating*. When ω_1 and ω_2 are close, the effect becomes more obvious:

$$\cos(\omega t) + \cos((\omega+\omega_b)t) \approx 2\cos(\omega t)\cos(\frac{\omega_b}{2}t), \quad (2)$$

where $\omega \gg \omega_b$. ω_b is called the beat frequency (see Figure 2).

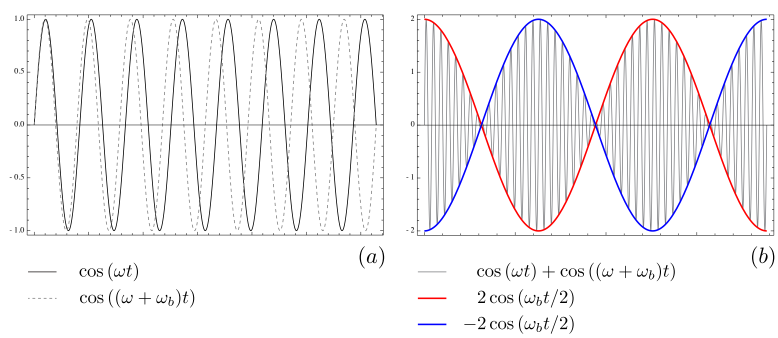

Figure 2: (a) two signals in a short interval, (b) beating phenomena

Note that the beating phenomena is quite robust under amplitude or phase noise (see Figure 3).

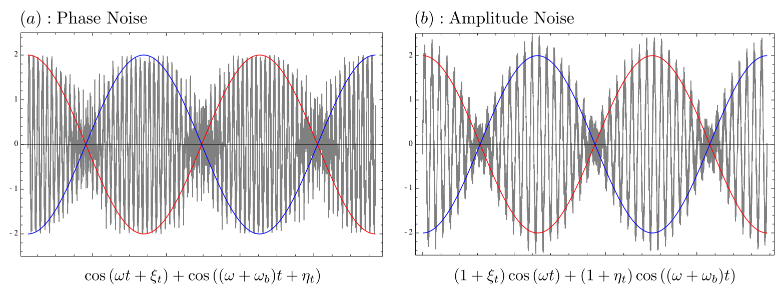

Figure 3: Beating is robust under both amplitude and phase noise. Here $\forall t,\ \ \xi_t, \eta_t \sim U(-1/2, 1/2)$

It is also straightforward to extend the result for multidimensional signals with different periodicities(Figure 4).

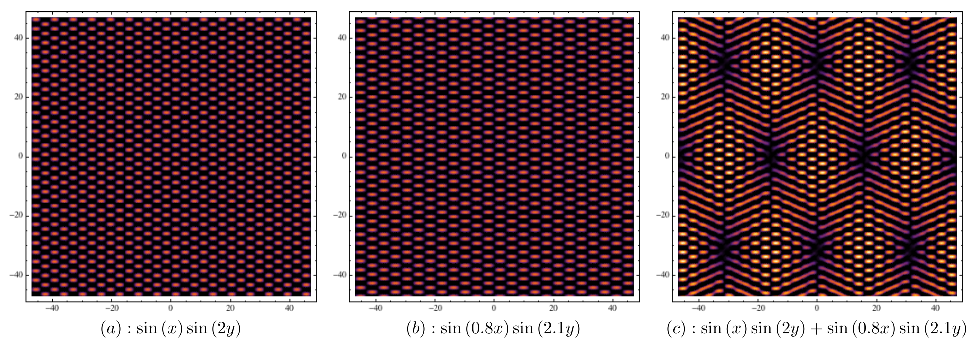

Figure 4: (a)Frequency in y axis, for the original signal is twice the frequency in x axis;(b) However, by adding a signal with changes in frequencies ($\omega_b^x = 2\omega_b^y$), the low frequency beats appear (c) note the two different beating frequencies in x and y direction

So far we have seen examples of beating in sinusoidal signals. To the best of our knowledge, no considerable research has been done on beating in arbitrary periodic signals. Here we show an example with spike trains which are of great interest in computational and systems neuroscience. Figure 5 demonstrates the beat phenomena for narrow gaussian spike trains where

$$\mathcal{S}_\omega^\sigma(t) = \sum_{i \in \mathbb{Z}} e^{-(t - i/\omega)^2/\sigma^2}. \quad (3)$$

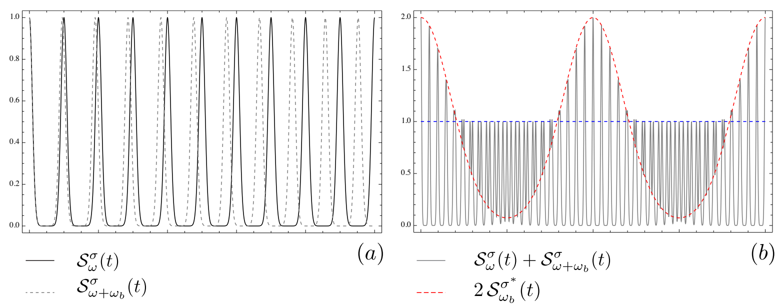

Figure 5: the beating phenomena for gaussian spike trains. Here we have approximated the envelope by $2\,\mathcal{S}^{\sigma^*}_{\omega_b}(t)$ where σ^* is estimated empirically. But the beating frequency is set theoretically and exactly matches the result.

2 McCulloch-Pitts Clock: a model for scaler variability

So after all, why is beating interesting? One of the important facts about beating is that the beating frequency can not be seen (directly) by studying the fourier transform of a beating signal. That is, fourier decomposition assumes a linear superposition of the sinusoidals instead of a modulation scheme (which is multiplicative, hence is nonlinear). Therefore one might take beating as a mathematical illusion or only a different representation which does not carry any extra information. But it turns out that beating frequency is an essential property of the signal when it passes through nonlinear systems.

Let us consider feeding two signals with close frequencies into the simplest nonlinear model in neuroscience, namely McCulloch-Pitts threshold-based Neuron (Figure 6). If we set the threshold to be the maximum value of the beating signal ($\theta = \max_{t\in\mathbb{R}}(w_1x_1 + w_2x_2)$), the output will be a spike train tuned exactly on beating frequency of two signals. The important point here is that the output frequency is independent of the individual input frequencies (which can be seen in fourier transform of the signals) and is only sensitive to the beating frequency.

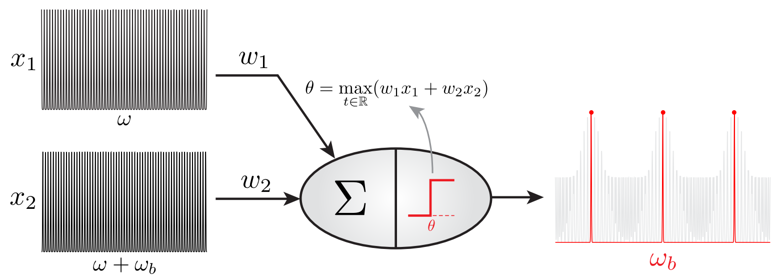

Figure 6: McCulloch-Pitts model: a beating-based tunable clock

Here the spectrum of the output signal consists of the beating frequency and its harmonics. The author believes that this phenomena is not well-studied in the field of computational neuroscience and may have a high potential to explain timing-related phenomena in neurobiological systems as well as providing a new way of thinking about pattern formation in nonlinear systems.

Let us assume that the brain, uses this simple model (McCulloch-Pitts Clock (MPC)) to create different time-scales. Now we can show that the accuracy of time-perception decreases linearly by increasing the length of the time intervals to be measured. To see that, let us consider a relaxed version of MPC with $\theta > \theta^*$ which passes a wider interval of values. To measure a single tick, the clock calculates the centre of mass of the passed interval (Figure 7). Obviously, for longer intervals, the width of the passed signal will linearly increase. Adding noise to the signals escalates the loss in accuracy while maintaining the linear relationship between the length of the interval and accuracy, however with a steeper slope.

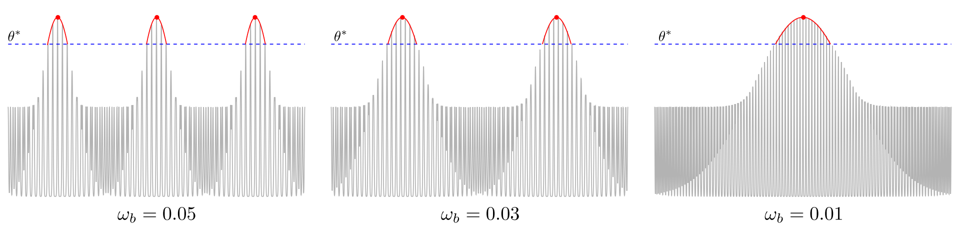

Figure 7: Ticking accuracy with MPC for different beating frequencies

Note that if we had considered the restricted MPC (as in Figure 6), by adding an amplitude or phase noise, we would still get the linear relationship between accuracy and length of the interval.

Let us summarize our observations:

- Experimental observations show a slight frequency difference in different brain cortical areas when the activity of the brain is quite oscillatory.
- Although from the fourier viewpoint, such differences are not usually taken seriously, they can cause low frequency beating phenomena.
- Beating is robust under noisy conditions.
- feeding a beating signal into a nonlinear filter (e.g. thresholding) shows that although beating frequency is not present in the spectrum of the signal, the result of applying a nonlinear filter is mostly influenced by it, rather than the actual frequency content of the signal.
- beating provides a consistent way for creating very different time scales.
- the accuracy of time measurement based on beating frequency linearly decreases by the length of the interval.

These observations can motivate a deeper look into the possible role of beating in neurobiological systems. In general, beating is not studied in a thorough manner in neuroscience community and further investigations may open new doors towards a better understanding of oscillatory systems in the brain. As for the timing problem, further experiments can be designed to assess the validity of the hypothesis.

On the theoretical side, I am trying to prove a similar mechanism for a much broader family of signals namely homogeneous poisson processes. I believe that adding up two poisson processes with close densities (λ), can be seen as a beating phenomena (a poisson process with $\lambda = |\lambda_1 - \lambda_2|$) when the resulting signal is clustered properly. Such a result may invoke more serious and broad implications in neuroscience and generally in the study of nonlinear and complex systems.